NATIONAL ADVISORY COMMITTEE
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TECHNICAL NOTE 2226

THEORETICAL ANALYSIS OF OSCILLATIONS IN HOVERING

OF HELICOPTER BLADES WITH INCLINED AND OFFSET

FLAPPING AND LAGGING HINGE AXES

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SUMMARY

The frequency and damping characteristics of the coupled flapping and lagging oscillations of helicopter blades in hovering are derived for the general case in which the lagging (vertical) hinge axis is off-set from the flapping hinge axis, while both hinge axes are inclined. An offset of the flapping hinge axis from the rotor axis of rotation is also considered. An inclination of the hinge axes causes pitch change of a blade section due to angular displacements about these axes. It is found from this analysis that an inclination of the flapping hinge axis chiefly affects the values of the natural flapping frequencies and the damping factors of the lagging motion of a blade. An inclination of the lagging hinge axis, which causes considerable flapping motion during the lagging oscillations, mainly affects the natural frequencies and damping factors in lagging. The analysis and the numerical examples worked out here indicate that significant increases in the damping of the lagging motions, which ordinarily border on instability, can be obtained by suitable inclinations of the hinge axes, especially of the lagging axis. Offsetting the flapping and lagging hinge axes tends especially to increase the natural lagging frequency.

INTRODUCTION

The aim of this investigation is to determine theoretically the characteristics of the rigid-body oscillations of helicopter rotor blades with inclined and offset flapping (horizontal) and lagging (vertical) hinge axes. For this purpose a simple method which requires the solution of a fourth-degree algebraic equation for obtaining the natural frequencies and damping decrements of the small coupled flapping and lagging oscillations of a blade in hovering about a state of static equilibrium is developed here. Approximate formulas for these characteristics, based on the quartic equation, are, moreover, given.

The equations of motion of a blade are derived by means of the Lagrange equations for a system having two degrees of freedom, namely, rotation about an inclined lagging hinge axis and rotation about an inclined flapping hinge axis. The flapping axis is assumed as offset from the rotor axis of rotation, while the lagging axis is further assumed as offset from the flapping axis. The aerodynamic loads in hovering are derived here on the basis of quasi-stationary flow, with the assumption of constant downwash throughout the rotor disk.

Basically, the effect of inclining either hinge axis is to create a kinematic relation between flapping or lagging and change of pitch of a blade section. This kinematic relation is of the form $\theta = f(\beta, \zeta)$, where θ, β, and ζ are, respectively, the pitch, flapping, and lagging angles. It may be noted that, although the effects of several kinematic constraints have already been derived in reference 1, the condition mentioned above has not been explicitly investigated there. Since the aerodynamic loads in hovering are largely functions of the blade, or pitch, angle, it follows that in the present analysis the effect of inclination of the hinge axes appears essentially in the resulting effect on the aerodynamic loads caused by the changes in blade angle.

The analysis is divided into three sections. In the first section the explicit equations of motion of a blade are derived. The equations, and their solutions, determining the steady-state (static in hovering) values of flapping, lagging, and blade angles are included here.

In the second section the dynamic equations are solved in general terms for the frequency and damping characteristics of the oscillations of a blade. Coupling between flapping and lagging, which becomes especially large when the lagging hinge axis is inclined, is taken into account. In addition to the development of a straightforward exact procedure, and explicit approximate solutions, for the natural frequencies and damping decrements of the oscillations, stability criterions are given. For any given data, these stability criterions, as well as the explicit equations for the details of the oscillation characteristics, can be readily applied. General conclusions regarding the effect of inclination and offset of the hinge axes are derived from the general solutions obtained.

In the final section numerical examples are worked out to illustrate the general conclusions reached as well as the method of application of the general solutions developed here.

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SYMBOLS

A, D	amplitude of flapping and lagging oscillations, respectively
a_0, a_1, a_2, a_3, a_4	coefficients of quartic equation (26a), defined by equations (26b)
c_{do}	profile-drag coefficient of blade section
c	chord length of blade section
c_0	value of c of innermost blade section
e_1	distance of flapping hinge axis from axis of rotation; flapping hinge eccentricity (fig. 1)
e_2	distance of lagging hinge axis from flapping hinge axis; lagging hinge eccentricity (fig. 1)
d_0	section profile drag per unit length of blade
d_{ox}, d_{oy}, d_{oz}	drag load components in x-, y-, and z-directions, respectively
$F_1, F_2, F_3, F_4,$ $F_5, F_6, F_7, F_8,$	dimensionless definite integrals defined by equations (21b)
g	acceleration due to gravity
$H = m_0/\rho \pi c_0 l$	
$i = \sqrt{-1}$	
$\bar{L}'$	lift loads based on quasi-stationary flow
L_1, L_2, L_3, L_4, L_5	dimensionless definite integrals defined by equations (21b)
L_x', L_y', L_z'	lift load components per unit blade length in x-, y-, and z-directions, respectively
l	length of blade measured from lagging hinge axis
$M = g/\Omega^2 l$	

m mass per unit length of blade

m_0 value of m at innermost blade section

n number of rotor blades

p complex frequency; if $p = \Omega(-R \pm i\omega)$ (R and ω real),
                             then $\omega\Omega/2\pi$ is the natural frequency in cycles
                             per second, while $2\pi R/\omega$ is the logarithmic decrement

p_{wt} unit weight loads of blade in z-direction

p_x, p_y, p_z aerodynamic plus weight loads per unit blade length
                             acting in x-, y-, and z-directions, respectively

Q_i generalized force

Q_β, Q_ζ generalized forces corresponding to generalized
                             coordinates β and ζ

q dimensionless complex frequency (p/Ω)

q_i generalized coordinate

q_f, q_l dimensionless flapping and lagging complex frequencies,
                             respectively

q_{fo}, q_{lo} dimensionless uncoupled flapping and lagging complex
                             frequencies, respectively

q_{f1}, q_{l1} first approximation to dimensionless coupled flapping
                             and lagging complex frequencies, respectively
                             (equations (28) and (30))

R tip radius of blade

r distance of any point on blade from axis z of
                             rotation

s coordinate along blade axis measured from lagging
                             hinge

s_1 value of s for innermost blade section

T kinetic energy of system

t time

$\overline{V}$ vector of resultant inflow velocity

V_{ns} resultant inflow velocity in plane of blade section

V_x, V_y, V_z components of resultant inflow velocity in hovering at any point of blade

W gross weight of helicopter

w_i induced downwash

X, Y, Z forces along axes of rotor system

$$\left(X = \int_{s_1}^{l} p_x \, ds; \quad Y = \int_{s_1}^{l} p_y \, ds; \quad Z = \int_{s_1}^{l} p_z \, ds \right)$$

x, y, z rotating right-handed Cartesian coordinate system in which z is axis of rotation and x coincides with undeflected longitudinal blade axis (figs. 1 and 2)

$$\dot{\bar{x}} = \dot{x} + (\Omega r)_x$$

$$\dot{\bar{y}} = \dot{y} + (\Omega r)_y$$

β_δ magnitude of angular displacement about inclined flapping hinge axis

β component of β_δ about a horizontal axis (AE in fig. 1) normal to deflected blade axis (OC)

$\bar{\beta}, \bar{\zeta}$ deviations of β and ζ from steady-state position

$\beta_o, \zeta_o, \theta_o$ steady-state values of β, ζ, and θ, respectively

$\bar{\Gamma}$ circulation vector

δ_1 inclination of lagging (vertical) hinge axis measured from a line parallel to z-axis (see fig. 2), positive when positive lagging (i.e., retreating) produces a decrease in blade angle

δ_3 inclination of flapping (horizontal) hinge axis measured from a line parallel to y-axis (see fig. 1), positive when positive flapping produces a decrease in blade angle

$\epsilon_1 = e_1 / l$

$\epsilon_2 = e_2/l$

ζ_δ magnitude of angular displacement about inclined lagging hinge axis

ζ component of ζ_δ about an axis (OD in fig. 2) parallel to z-axis (see fig. 1)

θ total pitch, or blade, angle of blade section after flapping and lagging deflections

θ_0 design, or initial, blade angle

$(\Delta\theta)_1$ change of pitch due to rotation ζ_δ about lagging hinge axis

$(\Delta\theta)_2$ change of pitch due to rotation β_δ about flapping hinge axis

$\xi = s/l$

$\xi_1 = s_1/l$

ρ density of air

ϕ angle between x-axis and radius of a point on blade to z-axis

Ω angular velocity of rotor system

$(\Omega r)_x, (\Omega r)_y$ components of linear rotational velocity of blade in x- and y-directions, respectively

ω natural frequency of oscillation of rotating blade

BASIC EQUATIONS

The basic equations of motion of a rotating helicopter blade in flapping and lagging are derived here.

Coordinate system.- A right-handed Cartesian coordinate system (x,y,z) in which the z-axis coincides with the axis of rotation of the rotor system and the x-axis is in the direction of the undeflected longitudinal blade axis is used. The x- and y-axes rotate with a blade about the z-axis. The flapping hinge axis of a blade, which is assumed to be in the plane of rotation (x,y), is considered as offset by a distance e_1 from the axis of rotation and inclined at an angle δ_3

to a line parallel to the y-axis (see fig. 1). The lagging hinge axis, which is assumed to be in the vertical (x,z) plane, is considered to be offset from the flapping hinge axis by a distance e_2 and inclined at an angle δ_1 to a line parallel to the z-axis (see fig. 2).

Lagrange equations.- The Lagrange equations of motion can be expressed in general as follows:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \tag{1}$$

where t is time, T is the kinetic energy of the system, q_i is a generalized coordinate, and Q_i is the corresponding generalized force. The dot indicates differentiation with respect to time.

The kinetic energy of a moving blade is given by

$$T = \frac{1}{2}\int_{s_1}^{l} m\left(\dot{\bar{x}}^2 + \dot{\bar{y}}^2 + \dot{z}^2\right) ds \tag{2a}$$

where

$$\left.\begin{array}{l} \dot{\bar{x}} = \dot{x} + (\Omega r)_x \\[2ex] \dot{\bar{y}} = \dot{y} + (\Omega r)_y \end{array}\right\} \tag{2b}$$

l is the length of the blade measured from the lagging hinge, s_1 is the distance of the innermost blade section from the lagging hinge, m is the mass per unit length of a blade, ds is a blade-length element, Ω is the angular velocity of the rotor system, r is the distance of any point of the blade from the axis of rotation, and $(\Omega r)_x$ and $(\Omega r)_y$ denote components of the linear rotational velocity of a blade element in the x- and y-directions, respectively.

The position of a blade after lagging and flapping can be determined as follows:

First consider a rotation of magnitude ζ_δ about the lagging hinge axis. This produces a change of pitch $(\Delta\theta)_1$ of magnitude

$$(\Delta\theta)_1 = -\zeta_\delta \sin \delta_1 \tag{3}$$

Moreover, the component ζ of rotation about an axis parallel to the z-axis (OD in fig. 2) has the magnitude

$$\zeta = \zeta_\delta \cos \delta_1 \qquad (4)$$

Let the blade now rotate with an angular displacement of β_δ about the inclined flapping axis. Then the additional pitch change $(\Delta\theta)_2$ due to this rotation is the component of rotation about the deflected blade axis (OC in fig. 1). Thus

$$(\Delta\theta)_2 = \beta_\delta \sin(\zeta - \delta_3) \qquad (5)$$

Moreover, the component β of rotation of the blade about a horizontal axis (AE in fig. 1) normal to the deflected blade axis (OC) after lagging is given by

$$\beta = \beta_\delta \cos(\delta_3 - \zeta) \qquad (6)$$

The coordinates of a point P on the longitudinal axis of a blade at a distance s (fig. 1) from the lagging hinge axis can be expressed in terms of the "flapping" and "lagging" displacement-components β and ζ by the following expressions:

$$\left.\begin{aligned} x &= e_1 + s \cos \beta \cos \zeta + e_2 \cos \frac{\beta \cos \delta_3}{\cos(\delta_3 - \zeta)} \\[2mm] y &= -s \cos \beta \sin \zeta \\[2mm] z &= s \sin \beta + e_2 \sin \frac{\beta \cos \delta_3}{\cos(\delta_3 - \zeta)} \end{aligned}\right\} \qquad (7)$$

The velocity components at any point of a blade which are due to the rotation of the rotor system can be expressed as follows (see fig. 3):

$$\left.\begin{aligned} (\Omega r)_x &= -\Omega r \sin \phi = -\Omega y \\[2mm] (\Omega r)_y &= \Omega r \cos \phi = \Omega x \end{aligned}\right\} \qquad (8)$$

where x and y are given by equations (7).

By considering β and ζ as the two generalized coordinates, equations (2a), (2b), (7), and (8) may be substituted into equation (1) to give the left side in terms of β, ζ, and time derivatives thereof.

Generalized forces.- The generalized forces Q_i here are due to the aerodynamic and weight loads. The force Q_i $\left(\text{here } Q_\zeta{}' \text{ and } Q_\beta\right)$ is defined by

$$Q_\zeta \, \delta\zeta + Q_\beta \, \delta\beta = X \, \delta x + Y \, \delta y + Z \, \delta z \qquad (9a)$$

where

$$\left.\begin{array}{l} X = \displaystyle\int_{s_1}^{l} p_x \, ds \\[20pt] Y = \displaystyle\int_{s_1}^{l} p_y \, ds \\[20pt] Z = \displaystyle\int_{s_1}^{l} p_z \, ds \end{array}\right\} \qquad (9b)$$

and p_x, p_y, and p_z are the aerodynamic plus weight loads per unit blade length in the x-, y-, and z-directions, respectively.

From equation (9a) it follows that

$$\left.\begin{array}{l} Q_\beta = X \dfrac{\partial x}{\partial \beta} + Y \dfrac{\partial y}{\partial \beta} + Z \dfrac{\partial z}{\partial \beta} \\[20pt] Q_\zeta = X \dfrac{\partial x}{\partial \zeta} + Y \dfrac{\partial y}{\partial \zeta} + Z \dfrac{\partial z}{\partial \zeta} \end{array}\right\} \qquad (10)$$

The derivatives $\dfrac{\partial x}{\partial \beta}$, $\dfrac{\partial x}{\partial \zeta}$, and so forth can be determined in terms of β and ζ by means of equations (7).

Lift loads.- The lift loads, based on quasi-stationary flow, can be found from the three-dimensional Kutta-Joukowski theorem:

$$\overline{L}' = \rho \overline{V} \times \overline{\Gamma} \qquad (11)$$

where ρ is the density of the fluid medium, $\bar{V}$ is the vector of the resultant inflow velocity, and $\bar{\Gamma}$ is the circulation vector. The velocity components in hovering at any point of a blade can be expressed as:

$$V_x = \Omega y - \dot{x}$$

$$V_y = -\Omega x - \dot{y} \qquad\qquad\qquad (12)$$

$$V_z = -w_i - \dot{z}$$

where w_i is the induced downwash at the rotor, while x, y, and z can be put in terms of β and ζ by means of equations (7). The induced downwash is here assumed as constant throughout the rotor disk in hovering. According to the simple momentum theory, it can be calculated from the expression:

$$\frac{w_i}{\Omega l} = \sqrt{\frac{W}{2\pi R^2 \rho \Omega^2 l^2}} \qquad\qquad\qquad (13)$$

where W is the gross weight of the helicopter and R is the tip radius of a blade.

By considering the circulation vector $\bar{\Gamma}$ to coincide with the deflected longitudinal blade axis and to have a magnitude determined by the Kutta condition of finite trailing-edge velocity in two-dimensional flow, the lift loads can be derived in a manner similar to that given in detail in reference 1. By assuming β, ζ, θ (the blade angle), and $w_i/\Omega l$ to be first order small, the following expressions for the lift load components per unit length of a blade can thus be derived to second-order quantities:

$$L_x' = \rho\pi c\Omega(e_1 + e_2 + s)\beta\left[\dot{\beta}(e_2 + s) + w_i - \Omega(e_1 + e_2 + s)\theta\right]$$

$$L_y' = \rho\pi c\left[\dot{\beta}(e_2 + s) + w_i\right]\left[\dot{\beta}(e_2 + s) + w_i - \Omega(e_1 + e_2 + s)\theta\right]$$

$$L_z' = \rho\pi c\left\{\left[\dot{\beta}(e_2 + s) + w_i\right]\left[\dot{\zeta}s - \Omega(e_1 + e_2 + s)\right] - 2\dot{\zeta}\Omega s(e_1 + e_2 + s)\theta - \right.$$

$$\left. \Omega^2(e_1 + e_2 + s)\left[\beta\zeta(e_1 + e_2) - (e_1 + e_2 + s)\theta\right]\right\} \qquad (14)$$

where c is the chord length of a blade section.

Drag loads.- The expressions for the lift load components include the effect of induced drag, since they include the effect of downwash. The profile-drag loads can be determined from the following expression for the profile drag per unit length of a blade:

$$d_o = c_{do} \frac{\rho}{2} V_{ns}^2 c \qquad (15)$$

where c_{do} is the profile-drag coefficient of a blade section and V_{ns} is the resultant inflow velocity in the plane of the blade section. By considering c_{do} as constant and the resultant drag force as acting parallel to the resultant inflow velocity, the following expressions, with c_{do} considered as first order small, are obtained for the drag load components to second-order quantities:

$$d_{ox} = -\frac{1}{2} \zeta \rho c c_{do} \Omega^2 s(e_1 + e_2 + s)$$

$$d_{oy} = \frac{1}{2} \rho c c_{do} \Omega(e_1 + e_2 + s)\left[2\dot\zeta s - \Omega(e_1 + e_2 + s)\right] \qquad (16)$$

$$d_{oz} = -\frac{1}{2} \rho c c_{do} \Omega(e_1 + e_2 + s)\left[w_1 + \dot\beta(e_2 + s)\right]$$

Weight loads.- The unit weight loads of a blade act in the z-direction and may be expressed in the form:

$$p_{wt} = -mg \qquad (17)$$

where g is the acceleration due to gravity.

Equations of motion.- The total pitch angle of a blade section after deflecting is, according to relations (3) to (6), given by:

$$\theta = \theta_0 + \beta \tan(\zeta - \delta_3) - \zeta \tan \delta_1 \qquad (18)$$

where θ_0 is the initial, or design, blade angle before flapping or lagging. This expression for θ must be substituted in equations (14) for the lift loads.

The equations of motion of a blade in flapping and lagging can now be written explicitly in terms of β and ζ by substituting into equation (1). By putting

$$\beta = \beta_0 + \overline\beta(t)$$

$$\zeta = \zeta_0 + \overline\zeta(t)$$

where β_0 and ζ_0 denote the values of β and ζ in the steady state, while $\beta(t)$ and $\overline{\zeta}(t)$ denote small deviations from the steady state, and observing that the steady-state values must alone satisfy the equations of motion, the following nondimensional equations are obtained for the values of β_0 and ζ_0 and for the vibrational deviations $\overline{\beta}(t)$ and $\overline{\zeta}(t)$:

$$\beta_0 H \int_{\xi_1}^{1} \frac{m}{m_0}(\epsilon_2 + \xi)(\epsilon_1 + \epsilon_2 + \xi)\, d\xi + MH \int_{\xi_1}^{1} \frac{m}{m_0}(\epsilon_2 + \xi)\, d\xi +$$

$$\int_{\xi_1}^{1} \frac{c}{c_0}(\epsilon_1 + \epsilon_2 + \xi)(\epsilon_2 + \xi)\left[(\epsilon_1 + \epsilon_2)\beta_0\zeta_0 + \right.$$

$$\left. \frac{w_i}{\Omega l}\left(1 + \frac{c_{do}}{2\pi}\right) - (\epsilon_1 + \epsilon_2 + \xi)\theta_0\right] d\xi = 0 \qquad (19a)$$

$$\zeta_0 = \frac{\displaystyle\int_{\xi_1}^{1} \frac{c}{c_0}\left[\frac{c_{do}}{2\pi}(\epsilon_1 + \epsilon_2 + \xi)^2 + \frac{w_i}{\Omega l}(\epsilon_1 + \epsilon_2 + \xi)\theta_0 - \left(\frac{w_i}{\Omega l}\right)^2\right]\xi\, d\xi}{\displaystyle H(\epsilon_1 + \epsilon_2)\int_{\xi_1}^{1} \frac{m}{m_0}\xi\, d\xi} \qquad (19b)$$

$$\frac{\dot{\overline{\zeta}}}{\Omega}(F_1 - HF_2) + \frac{\dot{\overline{\beta}}}{\Omega} F_3 + \frac{\ddot{\overline{\beta}}}{\Omega^2} HF_4 + \overline{\zeta}(F_5 + F_8 \tan\delta_1) + \overline{\beta}(HF_6 + F_7) = 0 \qquad (20a)$$

$$\frac{\dot{\overline{\beta}}}{\Omega}(HF_2 + L_1) + \frac{\dot{\overline{\zeta}}}{\Omega} L_2 + \frac{\ddot{\overline{\zeta}}}{\Omega^2} HL_3 + \overline{\beta}\left(\tan\delta_3 - \zeta_0 \sec^2\delta_3\right)L_4 +$$

$$\overline{\zeta}\left[HL_5 + \left(\tan\delta_1 - \beta_0 \sec^2\delta_3\right)L_4\right] = 0 \qquad (20b)$$

where

$$\left. \begin{array}{ll} \xi = \dfrac{s}{l} & \xi_1 = \dfrac{s_1}{l} \\[2ex] \epsilon_1 = \dfrac{e_1}{l} & \epsilon_2 = \dfrac{e_2}{l} \\[2ex] H = \dfrac{m_0}{\rho\pi c_0 l} & M = \dfrac{g}{\Omega^2 l} \end{array} \right\} \qquad (21a)$$

$$F_1 = \int_{\xi_1}^{1} \frac{c}{c_o}(\epsilon_2 + \xi)\xi\left[2(\epsilon_1 + \epsilon_2 + \xi)\theta_o - \frac{w_1}{\Omega l}\right] d\xi$$

$$F_2 = 2\beta_o \int_{\xi_1}^{1} \frac{m}{m_o}\xi(\epsilon_2 + \xi)\ d\xi$$

$$F_3 = \left(1 + \frac{c_{do}}{2\pi}\right)\int_{\xi_1}^{1} \frac{c}{c_o}(\epsilon_2 + \xi)^2(\epsilon_1 + \epsilon_2 + \xi)\ d\xi$$

$$F_4 = \int_{\xi_1}^{1} \frac{m}{m_o}(\epsilon_2 + \xi)^2\ d\xi$$

$$F_5 = \beta_o \int_{\xi_1}^{1} \frac{c}{c_o}(\epsilon_1 + \epsilon_2 + \xi)(\epsilon_2 + \xi)\left[\epsilon_1 + \epsilon_2 - (\epsilon_1 + \epsilon_2 + \xi)\ \sec^2\delta_3\right] d\xi$$

$$F_6 = \int_{\xi_1}^{1} \frac{m}{m_o}(\epsilon_2 + \xi)\left(\epsilon_1 + \epsilon_2 + \xi - M\beta_o\right) d\xi$$

$$F_7 = \int_{\xi_1}^{1} \frac{c}{c_o}(\epsilon_1 + \epsilon_2 + \xi)(\epsilon_2 + \xi)\left[(\epsilon_1 + \epsilon_2)\ \zeta_o + \right.$$

$$\left. (\epsilon_1 + \epsilon_2 + \xi)\left(\tan\delta_3 - \zeta_o\ \sec^2\delta_3\right)\right] d\xi$$

$$F_8 = \int_{\xi_1}^{1} \frac{c}{c_o}(\epsilon_1 + \epsilon_2 + \xi)^2(\epsilon_2 + \xi)\ d\xi$$

$$L_1 = \int_{\xi_1}^{1} \frac{c}{c_o}(\epsilon_2 + \xi)\xi\left[2\ \frac{w_1}{\Omega l} - (\epsilon_1 + \epsilon_2 + \xi)\theta_o\right] d\xi$$

$$L_2 = \frac{c_{do}}{\pi}\int_{\xi_1}^{1} \frac{c}{c_o}\xi^2(\epsilon_1 + \epsilon_2 + \xi)\ d\xi$$

$$L_3 = \int_{\xi_1}^{1} \frac{m}{m_o}\xi^2\ d\xi$$

$$L_4 = \frac{w_1}{\Omega l}\int_{\xi_1}^{1} \frac{c}{c_o}\xi(\epsilon_1 + \epsilon_2 + \xi)\ d\xi$$

$$L_5 = (\epsilon_1 + \epsilon_2)\int_{\xi_1}^{1} \frac{m}{m_o}\xi\ d\xi$$

$$\Biggr\}(21b)$$

m_0 and c_0 refer to the mass and chord, respectively, of the innermost blade section, and θ_0 is the steady-state value of θ. The value of θ_0 is given by:

$$\theta_0 = \theta_0 + \beta_0 \tan\left(\zeta_0 - \delta_3\right) - \zeta_0 \tan \delta_1 \tag{22}$$

In deriving equations (20a) and (20b) the vibrational deviations $\bar{\beta}$ and $\bar{\zeta}$ (not β_0 and ζ_0) have been assumed sufficiently small so that only first powers of $\bar{\beta}$ and $\bar{\zeta}$ may be retained.

The condition that the lift load support the gross weight of the helicopter in hovering leads to the following expression for θ_0:

$$\theta_0 = \frac{\dfrac{W}{n\Omega^2\rho\pi c_0 l} + \left[\dfrac{w_1}{\Omega l}\left(1 + \dfrac{c_{do}}{2\pi}\right) + (\epsilon_1 + \epsilon_2)\beta_0\zeta_0\right]\displaystyle\int_{\xi_1}^{1} \dfrac{c}{c_0}(\epsilon_1 + \epsilon_2 + \xi)\,d\xi}{\displaystyle\int_{\xi_1}^{1} \dfrac{c}{c_0}(\epsilon_1 + \epsilon_2 + \xi)^2\,d\xi} \tag{23}$$

where n is the number of blades in the rotor system.

Steady-state values of β, ζ, and θ.- The steady-state equations (19a), (19b), and (23) can be solved for the values of β_0, ζ_0, and θ_0 as follows. Equation (23) may be solved first for θ_0 with the higher-order term (in $\beta_0\zeta_0$) there neglected. The value of ζ_0 can then be obtained from equation (19b) and the value of β_0 can subsequently be determined from equation (19a). A second approximation for the value of θ_0 can now be calculated by inserting the values of β_0 and ζ_0, thus obtained, into equation (23). Second approximations for β_0 and ζ_0, by using the new value of θ_0, can then be determined from equations (19a) and (19b). It will usually be found that the second approximations in this iterative procedure are sufficiently accurate for practical purposes. The value of the design angle θ_0 can be found from the relation

$$\theta_0 = \theta_0 - \beta_0 \tan\left(\zeta_0 - \delta_3\right) + \zeta_0 \tan \delta_1$$

It may be noted here that the values of β_0, ζ_0, and θ_0 are all independent of the inclinations δ_1 and δ_3 of the hinge axes. However, the appropriate value of the design blade angle θ_0 does depend on the values of δ_1 and δ_3.

Dynamic equations - physical significance.- Equations (20a) and (20b) are the equations determining the characteristics of the small vibrations of a helicopter blade about the state of static equilibrium given, as explained above, by equations (19a) and (19b). The physical significance of each of the terms in equations (20a) and (20b) is as follows. The terms proportional to $\ddot{\beta}$ and $\ddot{\zeta}$ represent the inertia loads per unit of length acting on a blade. The term proportional to $\dot{\beta}$ in equation (20a) represents primarily the aerodynamic damping load caused by a change of angle of attack induced by a vertical velocity $\dot{\beta}$ of a blade element. The term $\bar{\beta} HF_6$ in equation (20a) is due to the centrifugal loads, while the term $\bar{\beta} F_7$ represents the lift load. The term $\frac{\dot{\zeta}}{\Omega} F_1$ in equation (20a) represents a lift load component (including the effect of downwash) caused by a lagging velocity $\dot{\zeta}$, while the term $\frac{\dot{\zeta}}{\Omega} HF_2$ represents a Coriolis load. The term proportional to $\bar{\zeta}$ in equation (20a) is a lift load.

In equation (20b), the term $\frac{\dot{\zeta}}{\Omega} L_2$ is proportional to the profile drag of a blade section. The term $\bar{\zeta} HL_5$ in this equation represents the centrifugal restoring force in lagging, while the term proportional to $\bar{\zeta} L_4$ represents an induced drag, due to the downwash w_i. The term $\frac{\dot{\beta}}{\Omega} HF_2$ represents a Coriolis load while the term $\frac{\dot{\beta}}{\Omega} L_1$ is an aerodynamic load. Finally, the term $\bar{\beta} L_4$ is an aerodynamic load due to the downwash w_i.

Equation (20a) corresponds primarily to the flapping oscillations, while equation (20b) corresponds primarily to the lagging oscillations. It will be observed, however, that these equations are coupled with each other. It may be further noted that all of the coupling terms in lagging (proportional to $\bar{\beta}$ and $\dot{\beta}$ in equation (20b)) have magnitudes which are at most first order small but that in flapping there is a finite-order coupling term proportional to $\tan \delta_1$. Consequently, inclination of the vertical (lagging) hinge tends to produce considerable coupling between flapping and lagging.

GENERAL SOLUTIONS FOR STABILITY AND VIBRATION CHARACTERISTICS

The general solution of the flapping and lagging equations and the general implications of this solution are given here. In addition, stability conditions are shown and discussed.

Basic solution.- Equations (20a) and (20b) are a system of linear, homogeneous, differential equations with constant coefficients. The solution of this system can be obtained by setting

$$\left.\begin{array}{c} \overline{\beta} = Ae^{pt} \\[2mm] \overline{\zeta} = De^{pt} \end{array}\right\} \tag{24}$$

where A and D are real or complex constants (amplitudes) and p is a real or complex constant ("frequency").

The following linear, simultaneous, algebraic equations in A and D are obtained by substituting equations (24) into equations (20a) and (20b):

$$A\left(q^2 HF_4 + qF_3 + HF_6 + F_7\right) + D\left[q\left(F_1 - HF_2\right) + F_5 + F_8 \tan \delta_1\right] = 0 \tag{25a}$$

$$A\left[q\left(HF_2 + L_1\right) + \left(\tan \delta_3 - \zeta_0 \sec^2\delta_3\right)L_4\right] +$$

$$D\left[q^2 HL_3 + qL_2 + HL_5 + \left(\tan \delta_1 - \beta_0 \sec^2\delta_3\right)L_4\right] = 0 \tag{25b}$$

where $q \equiv p/\Omega$.

For the existence of a nontrivial solution to the above equations the determinant of the coefficients of this system must vanish. Expanding this determinant and setting it equal to zero leads to the following quartic equation (to second powers of small-order quantities) for the complex dimensionless frequency q:

$$a_4 q^4 + a_3 q^3 + a_2 q^2 + a_1 q + a_0 = 0 \tag{26a}$$

where

$$a_4 = H^2 F_4 L_3$$

$$a_3 = H\left(F_4 L_2 + F_3 L_3\right)$$

$$a_2 = HF_4\left[HL_5 + \left(\tan \delta_1 - \beta_0 \sec^2\delta_3\right)L_4\right] + HL_3\left(HF_6 + F_7\right) + F_3 L_2 +$$

$$\left(HF_2 - F_1\right)\left(HF_2 + L_1\right)$$

$$a_1 = F_3\left[HL_5 + \left(\tan \delta_1 - \beta_0 \sec^2\delta_3\right)L_4\right] + L_2\left(HF_6 + F_7\right) +$$

$$\left(HF_2 - F_1\right)L_4 \tan \delta_3 - \left(HF_2 + L_1\right)\left(F_5 + F_8 \tan \delta_3\right)$$

$$a_0 = \left(HF_6 + F_7\right)\left[HL_5 + \left(\tan \delta_1 - \beta_0 \sec^2\delta_3\right)L_4\right] +$$

$$L_4\left[F_8 \tan \delta_1\left(\zeta_0 \sec^2\delta_3 - \tan \delta_3\right) - F_5 \tan \delta_3\right]$$

$$\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} \quad (26b)$$

Frequency and damping characteristics.- Equation (26a) determines four values of q. If these values constitute two pairs of complex conjugate roots, then they can be interpreted physically as follows.

If $q = -R \pm i\omega$, where R and ω are real positive quantities, then $\omega\Omega$ is the natural frequency of oscillation, while $2\pi\frac{R}{\omega}$ is the logarithmic decrement, that is, the natural logarithm of the ratio of amplitude of oscillation at any time to the amplitude one period or cycle later. If R is negative, then the oscillations will increase in amplitude with time and the motion will be unstable. A negative real root for q indicates a nonoscillating subsident motion, which is stable, while a positive real root indicates a nonoscillating motion of steadily increasing amplitude, which is unstable.

Equation (26a) can be solved exactly for the values of q by means, for example, of Ferrari's method (reference 2, pp. 51 and 52). For purposes of a quicker, but approximate, calculation, the following method may be used.

Let a first approximation to a pair of roots of equation (26a) be
that which would be obtained in flapping if there is no coupling with
lagging. If one of these roots is denoted by q_{fo}, then it follows from
equation (25a) with $D = 0$ that:

$$q_{fo} = \frac{-F_3}{2HF_4} \pm \sqrt{\left(\frac{F_3}{2HF_4}\right)^2 - \frac{HF_6 + F_7}{HF_4}} \qquad (27)$$

A second approximation q_{f1} to equation (26a), by using Newton's method
(reference 2, pp. 96-97) with q_{fo} as first approximation whether it is
real or complex, will then be:

$$q_{f1} = q_{fo} + \frac{\left[q_{fo}\left(F_1 - HF_2\right) + F_5\right]\left[q_{fo}\left(HF_2 + L_1\right) + L_4 \tan \delta_3\right] + F_8 \tan \delta_1\left[q_{fo}\left(HF_2 + L_1\right) + \left(\tan \delta_3 - \zeta_0 \sec^2\delta_3\right)L_4\right]}{\left(q_{fo}^2 HL_3 + q_{fo}L_2 + HL_5 + L_4 \tan \delta_1\right)\left(2q_{fo}HF_4 + F_3\right) - F_8\left(HF_2 + L_1\right) \tan \delta_1} \qquad (28)$$

Similarly, a first approximation to the second pair of roots q_{lo}
of equation (26a), obtained by considering lagging without coupling to
flapping (equation (25b) with $A = 0$), may be assumed as

$$q_{lo} = -\frac{L_2}{2HL_3} \pm \sqrt{\left(\frac{L_2}{2HL_3}\right)^2 - \frac{HL_5 + \left(\tan \delta_1 - \beta_0 \sec^2\delta_3\right)L_4}{HL_3}} \qquad (29)$$

A second approximation, by Newton's method, will be

$$q_{l1} = q_{lo} + \frac{\left[q_{lo}\left(F_1 - HF_2\right) + F_5\right]\left[q_{lo}\left(HF_2 + L_1\right) + L_4 \tan \delta_3\right] + F_8 \tan \delta_1\left[q_{lo}\left(HF_2 + L_1\right) + \left(\tan \delta_3 - \zeta_0 \sec^2\delta_3\right)L_4\right]}{\left(q_{lo}^2 HF_4 + q_{lo}F_3 + HF_6 + F_7\right)\left(2q_{lo}HL_3 + L_2\right) - F_8\left(HF_2 + L_1\right) \tan \delta_1} \qquad (30)$$

Equations (27) and (29) may be interpreted physically as giving
the oscillation characteristics in flapping and lagging, respectively,
if there is no coupling between these motions. Equations (28) and (30)
give a first approximation to the effect of the coupling on the damping
and frequency characteristics of the corresponding motions. If the
corrections indicated by equations (28) and (30) are found to be rela-
tively large, then it is suggested that the roots of equation (26a) be
calculated by Ferrari's method.

Corresponding to any root of equation (26a), equation (25a) or
equation (25b) determines the value of the ratio A/D, whose absolute
value gives the ratio of the flapping amplitude to the lagging ampli-
tude. If there is no coupling, then $\left|\frac{A}{D}\right| = \infty$ for flapping, while
$\left|\frac{A}{D}\right| = 0$ for lagging. A complex value for A/D indicates a phase
difference between the component flapping and lagging motions.

From equations (27) and (28), it will be found that in ordinary cases the angle of inclination δ_1 of the lagging hinge axis has relatively little effect on the damping and frequency characteristics of the motion of a blade corresponding primarily to flapping. The angle of inclination δ_3 of the flapping hinge axis has, however, an important effect on the frequency characteristics of the primarily flapping motion of a blade. Increasing δ_3 negatively diminishes the natural frequency in flapping, since the total effective restoring force (proportional to $\overline{\beta}$ in equation (20a)) is diminished. It may be noted that a negative δ_3 indicates an inclination of the flapping hinge axis such that a positive (upward) flapping motion produces an increase in the blade angle θ (cf. equation (5)). For sufficiently large negative values of δ_3, in fact, the restoring force becomes negative and an unstable motion results, evidenced by at least one positive real root for q. Physically, the reason for the decrease of restoring force in flapping with increasing negative values of δ_3 is that a negative δ_3 tends to produce an aerodynamic (lift) load component opposing the centrifugal restoring force.

From equations (29) and (30) it will be found that increasing the angle δ_1 positively[1] increases the natural frequency of the motion corresponding primarily to lagging but decreases the damping factor (i.e., tends to make the real part of q positive). The natural frequency of the primarily lagging motion is only slightly affected by the angle δ_3. However, the damping factor in lagging is considerably influenced by δ_3 (cf. numerical examples discussed in a following section).

From equations (27) and (29), as well as equations (28) and (30), it will be found that, in general, oscillating motions corresponding to flapping will be much more heavily damped than those corresponding to lagging, since the negative real parts of q_F will be of higher order of magnitude than those of q_l.

The effect of offset, expressed quantitatively by the parameters ϵ_1 and ϵ_2, is included in the general solutions given here. It will be found from equation (29), in fact, that the chief effect of small offsets is an increase in the uncoupled lagging frequency with increase in the value of $\epsilon_1 + \epsilon_2$.

The coupling effects, as given approximately by equations (28) and (30), are much more important in the lagging motions (equation (30))

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[1]A positive value of δ_1 means that the lagging hinge axis is inclined such that a lagging (retreating) motion of the blade produces a decrease in the blade angle θ (cf. equation (3)).

than in the flapping motions (equation (28)). This is due to the fact
that, although the corrections due to coupling in both of these equa-
tions are at most first order small, the magnitude of the uncoupled
lagging complex frequency q_{l_0} (especially its real part) is also small;
consequently, the correction in lagging is relatively important. The
relatively important effect of coupling in lagging is also evidenced by
the relatively large value of $\left|\frac{A}{D}\right|$ (order of magnitude of unity) corre-
sponding to q_l when δ_1 is not zero.

General stability criterions.- If one is interested only in the
stability of the motion of a helicopter blade, then it is not necessary
to determine the actual values of the four roots of equation (26a).

The general condition that the motion be stable is that all of the
coefficients of equation (26a) have the same sign (viz, positive), while
also (reference 3)

$$a_1\left(a_2 a_3 - a_1 a_4\right) - a_0 a_3^2 > 0 \qquad (31a)$$

Since a_4 and a_3 are necessarily positive, it is necessary for sta-
bility that

$$\left.\begin{array}{l} a_2 > 0 \\[2ex] a_1 > 0 \\[2ex] a_0 > 0 \end{array}\right\} \qquad (31b)$$

Conditions (31a) and (31b) together are necessary and sufficient for
stability.

For purposes of a quick, but rough, calculation, the stability
criterions for the uncoupled flapping motions may be considered. From
expression (27) for q_{f_0}, this condition is found to be approximately:

$$\tan \delta_3 > \zeta_0 \sec^2 \delta_3 - \frac{H \displaystyle\int_{\xi_1}^{1} \frac{m}{m_0}\left(\epsilon_2 + \xi\right)\left(\epsilon_1 + \epsilon_2 + \xi\right) d\xi}{\displaystyle\int_{\xi_1}^{1} \frac{c}{c_0}\left(\epsilon_2 + \xi\right)\left(\epsilon_1 + \epsilon_2 + \xi\right)^2 d\xi} \qquad (32)$$

In actual cases, this condition will usually be satisfied for $\delta_3 > - 45°$, approximately. Condition (32) may be considered as an approximation even with coupling taken into account, since, as noted above, the effect of coupling is relatively small in flapping. It must be observed, however, that inequality (32) is only a necessary but not a sufficient condition for stability of the entire motion of a blade, since it does not include the motions corresponding primarily to lagging.

NUMERICAL EXAMPLE

A numerical example is given here to indicate the type of results which may be obtained in practice and to illustrate the method outlined in the previous section for finding the oscillation characteristics of a rotor-blade system. The inclination of the lagging hinge axis δ_1 will first be permitted to assume a range of values, while the flapping hinge axis will not be inclined; that is, $\delta_3 = 0°$. Then the flapping hinge axis will be inclined at various angles δ_3 while the lagging hinge axis inclination δ_1 is fixed at $0°$. Finally, the case of inclining both hinge axes simultaneously will be considered.

The following data for a typical helicopter are assumed:

Gross weight of helicopter, W, pounds 3000
Number of rotor blades, n 3
Angular speed of rotor system, Ω, radians per second 25
Tip radius of a rotor, R, feet 21.5
Length of blade proper, l, feet 20
Length of eccentricity, e_1, feet 1
Length of eccentricity, e_2, feet 0.5
Chord width at root, c_0, feet 1
Airfoil section . . . . Clark Y
Profile-drag coefficient, c_{do} 0.01

The structure of the blade consists of a tubular steel spar with plywood ribs flanged on and plywood skin attached to them. The diameter of the spar at the root is 0.15 foot and the wall thickness is 3/16 inch. The effective area of the wood is assumed as four-tenths that of the steel, and the density of steel is 15.12 slugs per cubic foot while the density of wood is 1.5 slugs per cubic foot. It will be further assumed that the blade has a constant cross section. Therefore,

$$\frac{c}{c_0} = \frac{m}{m_0} = 1$$

It is necessary to determine the values of H, M, and $w_i/\Omega l$ first in order to obtain the steady-state values of β, ζ, and θ. By substituting into equations (13) and (21a) one obtains

$$H = 0.774014$$

$$M = 0.002576$$

$$\frac{w_i}{\Omega l} = 0.041665$$

The values of β_0, ζ_0, and θ_0 can now be found by the iterative procedure outlined in the section "Basic Equations." The following are the values obtained by approximately solving (i.e., neglecting higher-order terms in β_0 and ζ_0) equations (23), (19b), and (19a), respectively:

$$\theta_0 = 0.122583$$

$$\zeta_0 = 0.051955$$

$$\beta_0 = 0.070968$$

A second approximation for θ_0 is now obtained by substituting the above values of β_0 and ζ_0 into equation (23) and now retaining second-order terms. By using this new value of θ_0, second approximations for ζ_0 and β_0 are obtained from equations (19b) and (19a), respectively. The following are the values of the second approximations:

$$\theta_0 = 0.122969$$

$$\zeta_0 = 0.052162$$

$$\beta_0 = 0.071369$$

It can be seen that the second approximations here give sufficient accuracy, since they differ little from the first approximations.

It is now possible to calculate the parameters in terms of δ_3 given by equations (21b). Thus, it is found that

$$F_1 = 0.055504 \qquad F_5 = 0.002062 - 0.022350 \sec^2\delta_3$$

$$F_2 = 0.049364 \qquad F_6 = 0.385102$$

$$F_3 = 0.294369 \qquad F_7 = 0.001507 + 0.313161 \tan \delta_3 - 0.016335 \sec^2\delta_3$$

$$F_4 = 0.358958 \qquad F_8 = 0.313162$$

$$L_1 = -0.006139 \qquad\qquad L_3 = 0.333333$$

$$L_2 = 0.000875 \qquad\qquad L_4 = 0.015451$$

$$L_5 = 0.037500$$

The values of the uncoupled complex frequencies in flapping and lagging $\left(q_{fo}\right.$ and q_{lo}, respectively$\left.\right)$ are first obtained by solving equations (27) and (29), respectively. Then the first approximation to the coupled frequencies $\left(q_{f1}\right.$ and q_{l1}, respectively$\left.\right)$ of the coupled oscillations is found from equations (28) and (30). In addition, the quartic equation (26a) has been solved for q by Ferrari's (exact) method to test the accuracy of Newton's method as it is applied here for finding the approximate values of the coupled complex frequencies. Finally, the ratio of the flapping amplitude to the lagging amplitude A/D can be found from either equation (25a) or (25b). It should be mentioned here that the ratio A/D serves as a check, since a particular value of q should yield the same result from both equations (25a) and (25b). It is sometimes necessary to adjust the root slightly in order to satisfy both equations.

Tables I and II contain the values of q_{fo}, q_{f1}, q_f (exact), and A/D, and q_{lo}, q_{l1}, q_l (exact), and A/D, respectively, when δ_1 has a range of values and $\delta_3 = 0°$. Tables III and IV contain the same information for $\delta_1 = 0°$ and a range of values for δ_3. The results given in tables V and VI were obtained by inclining the lagging and flapping hinge axes simultaneously.

The results of the numerical examples given in tables I to VI illustrate the general conclusions reached in the preceding section. For example, the negative real parts of the flapping complex frequencies are in all cases much greater than those of the lagging complex frequencies, an indication of much greater damping of the flapping motions than of the lagging motions. Table II illustrates how increasing δ_1

algebraically adversely affects the stability of the lagging motions, since the real part of q_l becomes less and less negative and even positive. Increasing δ_1 negatively, on the other hand, has a beneficial effect on the damping of the lagging motions. From tables I and II, in fact, it may be seen that the case $\delta_3 = 0^o$, $\delta_1 = -45^o$ is quite satisfactory from a stability viewpoint, since the damping factor for the lagging oscillations is over ten times as great as when $\delta_1 = 0^o$, while the flapping oscillations remain stable. The increase of the natural lagging frequency (imaginary part of q_l) with algebraically increasing δ_1 is also shown here. As evidenced by the values of A/D in table II, considerable flapping motions accompany the lagging motions, unless $\delta_1 \approx 0^o$. Table I indicates the rather small effect which an inclination δ_1 of the lagging hinge axis has on the flapping oscillation characteristics.

Table III illustrates the increase of flapping frequency with algebraic increase of the angle δ_3. It also shows the danger of instability when δ_3 has an algebraically low value. Here, for $\delta_3 = -45^o$, the positive real root for q_f indicates instability. Table IV shows the relatively little influence of the inclination δ_3 of the flapping hinge axis on the natural lagging frequencies. It is interesting to note here, however, that inclining the flapping hinge axis at an angle of $\delta_3 = -30^o$ has a beneficial effect on the stability characteristics of the lagging motion, since the real part of q_l then has a negative value of at least three times that of q_l corresponding to the other values of δ_3 considered here. However, it may be observed from table III that, in this case, the flapping frequency (imaginary part of q_f) would then become dangerously low.

Of the three cases considered in tables V and VI $\left(\delta_1 \neq 0^o, \delta_3 \neq 0^o\right)$, the case $\delta_1 = -30^o$, $\delta_3 = 30^o$ appears to be the most suitable from the point of view of stability characteristics. The case of $\delta_1 = 30^o$, $\delta_3 = -30^o$ evidently leads to instability, since the real part of q_l is positive. Moreover, the case $\delta_1 = -30^o$, $\delta_3 = -30^o$ gives a slight instability in flapping (q_f has a small positive real value).

A comparison in the tables between corresponding values of q_{f1} with q_f, and q_{l1} with q_l, shows that in most of the numerical cases treated here, Newton's method (equations (28) and (30)) gives fairly good second[2] approximations for the complex roots of

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[2]The first approximations, namely the uncoupled complex frequencies, are given by equations (27) and (29).

equation (26a). The only cases here in which this second approximation gave unreliable results are: $\delta_1 = 0°$, $\delta_3 = -45°$ and $\delta_1 = -30°$, $\delta_3 = -30°$. The effect of coupling on the complex frequencies can be seen from the tables by comparing q_{f_0} with q_{f_1} or q_f, and q_{l_0} with q_{l_1} or q_l.

CONCLUSIONS

From an investigation to determine theoretically the characteristics of the rigid-body oscillations in hovering of helicopter blades with inclined and offset flapping and lagging hinge axes, the following conclusions are drawn:

1. The frequency and damping characteristics of the (in general, coupled) flapping and lagging oscillations of a helicopter blade can be calculated directly either by solving a quartic equation for the dimensionless complex frequency or by using a second approximation for the roots of this quartic.

2. From the solutions it is found that the main effect of inclining negatively the flapping (horizontal) hinge axis is to diminish the natural flapping frequency by diminishing the total restoring forces in flapping.

3. Inclination of the flapping hinge axis also has an important effect on the damping characteristics of the lagging motion, as evidenced by the numerical example treated here, which indicated an inclination of $-30°$ to be a desirable value from this viewpoint. This value, however, gave a dangerously low natural flapping frequency.

4. The chief effect of inclining negatively the lagging (vertical) hinge axis is to decrease the natural lagging frequency but to increase the damping factor, and therefore the stability, in lagging. Inclination of the lagging hinge axis increases the coupling between lagging and flapping and leads to large flapping motions accompanying the lagging motions.

5. The main effect of a slight offset of the hinge axes is an increase in the natural lagging frequencies.

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    Brooklyn, N. Y., October 24, 1949

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TABLE I.- VALUES OF q_{fo}, q_{f1}, q_f (EXACT), AND A/D

WHEN δ_1 HAS A RANGE OF VALUES AND $\delta_3 = 0°$

δ_1 (deg)	q_{fo} (equation (27))	q_{f1} (equation (28))	q_f (exact)	A/D
45	-0.5297 ± 0.8596i	-0.6388 ± 0.9323i	-0.5858 ± 0.9038i	5.181 ∓ 6.240i
30	-.5297 ± 0.8596i	-.5677 ± 0.8827i	-.5630 ± 0.8816i	4.904 ∓ 6.187i
0	-.5297 ± 0.8596i	-.5338 ± 0.8484i	-.5255 ± 0.8515i	4.470 ∓ 6.127i
-30	-.5297 ± 0.8596i	-.4878 ± 0.8249i	-.4795 ± 0.8241i	3.888 ∓ 6.217i
-45	-.5297 ± 0.8596i	-.4999 ± 0.8311i	-.4392 ± 0.8091i	3.514 ∓ 6.143i

TABLE II.- VALUES OF q_{lo}, q_{l1}, q_l (EXACT), AND A/D

WHEN δ_1 HAS A RANGE OF VALUES AND $\delta_3 = 0°$

δ_1 (deg)	q_{lo} (equation (29))	q_{l1} (equation (30))	q_l (exact)	A/D
45	-0.00169 ± 0.4100i	0.05671 ± 0.3753i	0.05435 ± 0.3845i	-0.9307 ± 0.4228i
30	-.00169 ± 0.3779i	.03117 ± 0.3638i	.03151 ± 0.3660i	-.5346 ± 0.2140i
0	-.00169 ± 0.3290i	-.005491 ± 0.3318i	-.005891 ± 0.3316i	.06312 ∓ 0.04713i
-30	-.00169 ± 0.2714i	-.04303 ± 0.2576i	-.04303 ± 0.2576i	.7447 ∓ 0.2425i
-45	-.00169 ± 0.2199i	-.07733 ± 0.2052i	-.07733 ± 0.2051i	1.302 ∓ 0.2896i

TABLE III.- VALUES OF q_{fo}, q_{f1}, q_f (EXACT), AND A/D

WHEN δ_3 HAS A RANGE OF VALUES AND $\delta_1 = 0°$

δ_3 (deg)	q_{fo} (equation (27))	q_{f1} (equation (28))	q_f (exact)	A/D
45	-0.5297 ± 1.344i	-0.5271 ± 1.339i	-0.5271 ± 1.339i	7.881 ∓ 9.042i
30	-.5297 ± 1.170i	-.5273 ± 1.165i	-.5274 ± 1.165i	6.444 ∓ 8.287i
0	-.5297 ± 0.8596i	-.5338 ± 0.8484i	-.5255 ± 0.8515i	4.470 ∓ 6.128i
-30	-.5297 ± 0.2617i	-.4695 ± 0.1974i	-.5093 ± 0.1809i	3.478 ∓ 1.033i
-45	.1389 -1.198	.1817 -1.222	.1737 -1.221	3.024 7.302

TABLE IV.- VALUES OF q_{lo}, q_{l1}, q_l (EXACT), AND A/D

WHEN δ_3 HAS A RANGE OF VALUES AND $\delta_1 = 0°$

δ_3 (deg)	q_{lo} (equation (29))	q_{l1} (equation (30))	q_l (exact)	A/D
45	-0.001696 ± 0.32241	-0.004358 ± 0.32991	-0.004360 ± 0.32981	0.07376 ∓ 0.023361
30	-.001696 ± 0.32681	-.004068 ± 0.33121	-.004165 ± 0.33111	.05907 ∓ 0.026801
0	-.001696 ± 0.32901	-.005491 ± 0.33181	-.005891 ± 0.33161	.06312 ∓ 0.047131
-30	-.001696 ± 0.32681	-.02232 ± 0.33451	-.02202 ± 0.33701	.08841 ∓ 0.24081
-45	-.001696 ± 0.32241	-.04357 ± 0.32241	-.007629 ± 0.35971	.2364 ∓ 0.21861

TABLE V.- VALUES OF q_{fo}, q_{f1}, q_f (EXACT), AND A/D

WHEN $\delta_1 \neq 0°$ AND $\delta_3 \neq 0°$

δ_1 (deg)	δ_3 (deg)	q_{fo} (equation (27))	q_{f1} (equation (28))	q_f (exact)	A/D
30	-30	-0.5297 ± 0.26171	-0.5471 ± 0.55121	-0.6048 ± 0.40481	4.093 ∓ 2.4631
-30	30	-.5297 ± 1.1701	-.5073 ± 1.1441	-.5055 ± 1.1491	6.206 ∓ 8.3721
-30	-30	-.5297 ± 0.26171	-.3374 ± 0.012681	$\begin{cases} -.8179 \\ .01907 \end{cases}$	1.997 5.260

TABLE VI.- VALUES OF q_{lo}, q_{l1}, q_l (EXACT), AND A/D

WHEN $\delta_1 \neq 0°$ AND $\delta_3 \neq 0°$

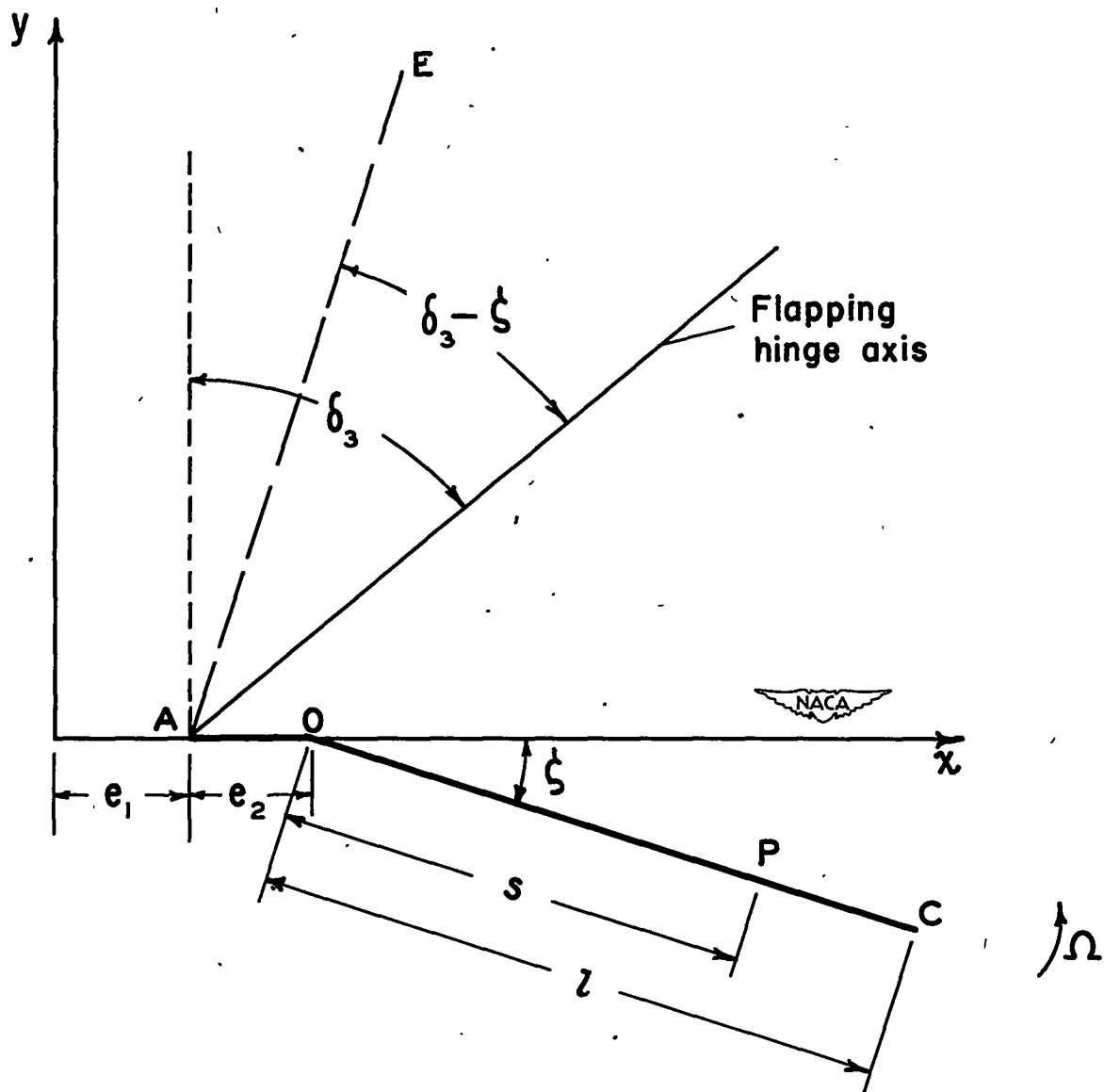| δ_1 (deg)	δ_3 (deg)	q_{lo} (equation (29))	q_{l1} (equation (30))	q_l (exact)	A/D
30	-30	-0.001696 ± 0.37601	0.07318 ± 0.34081	0.07341 ± 0.35891	-0.6374 ± 0.83611
-30	30	-.001696 ± 0.26871	-.02806 ± 0.29891	-.02598 ± 0.29951	.4698 ± 0.10461
-30	-30	-.001696 ± 0.26871	-.2173 ± 0.18201	-.1320 ± 0.45221	.04182 ∓ 2.1061

NACA

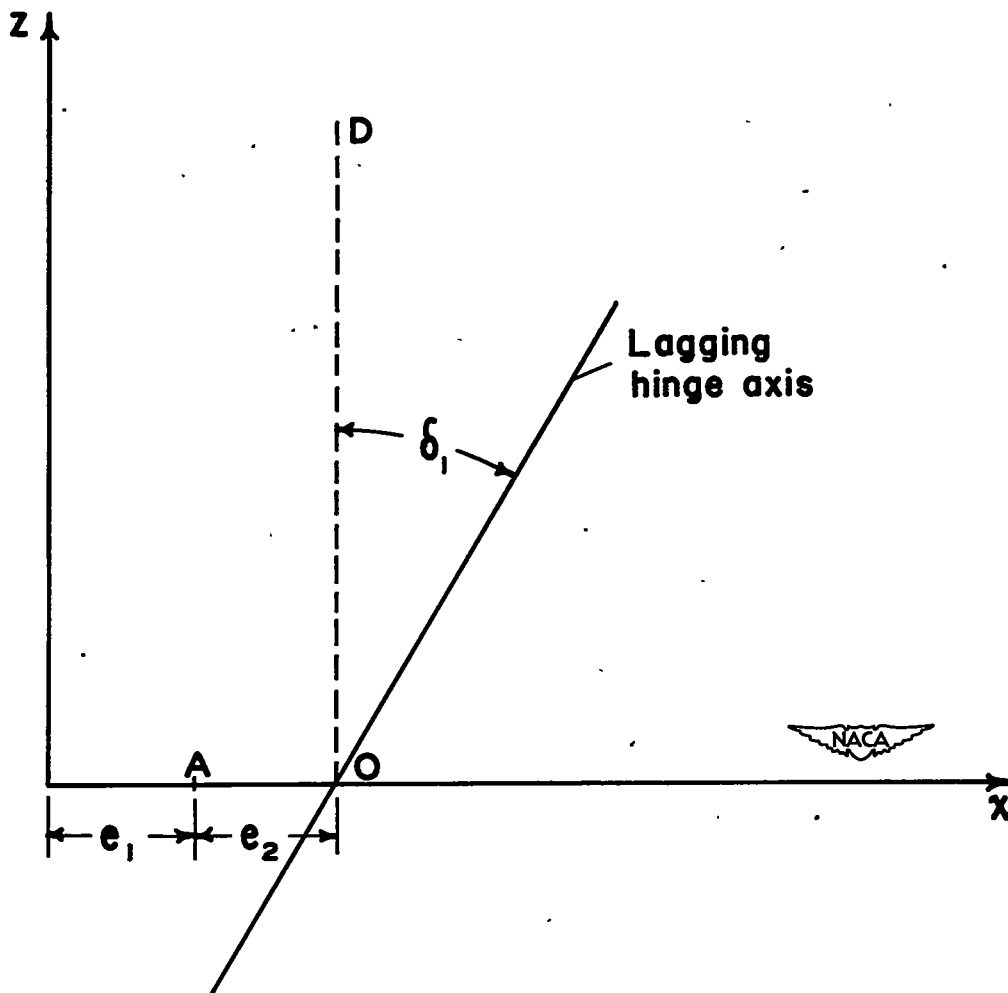Figure 1.- Blade configuration in plane (x,y) of rotation.
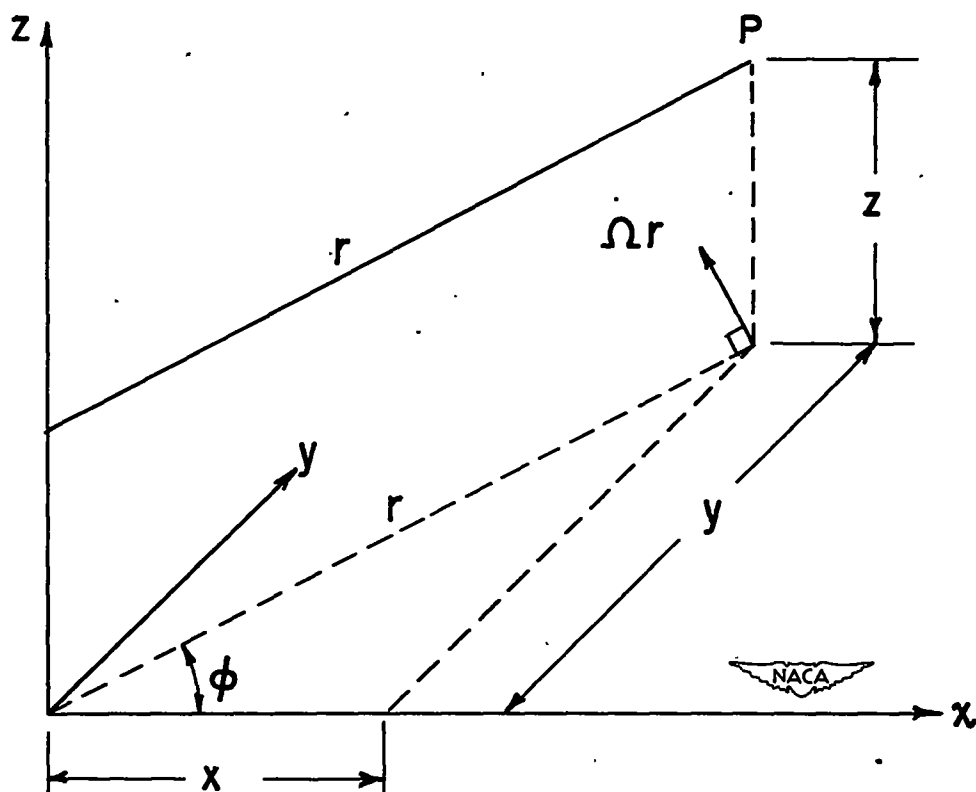
Figure 2.- Blade configuration in vertical (x, z) plane.

Figure 3.- Rotational velocity Ωr at a point of a blade (three-dimensional view).